Wednesday, January 28, 2015 7:37 AM

So what does $T^{\mu\nu}$ look like? $T^{\mu\nu} = \left(\begin{array}{c|ccc} T^{00} & T^{01} & T^{02} & T^{03} \\ \hline T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{array}\right)$ Use a matrix representation to organize our thoughts ☺!

First $T^{\mu\nu} = T^{\nu\mu}$ so we only need to evaluate ten components.

Consider: $n_\mu = (\overset{n_0=1}{1}, \underbrace{0, 0, 0}_{n_i=0})$ $dV = dx\,dy\,dz$ $p^\mu = \begin{pmatrix} E_{rel} \\ \vec{p}_{rel} \end{pmatrix}$

Then: $dP^\mu = T^{\mu\nu} n_\nu \, dV$

$dP^\mu = T^{\mu 0} n_0 \, dV$

$dP^\mu = T^{\mu 0} dV \Rightarrow \begin{cases} dP^0 = T^{00} dV & \Rightarrow T^{00} = \frac{dE}{dV} = \text{energy density} = \rho \\ dP^i = T^{i0} dV & \Rightarrow T^{i0} = \frac{dp^i}{dV} = \text{momentum density} = \pi^i \end{cases}$

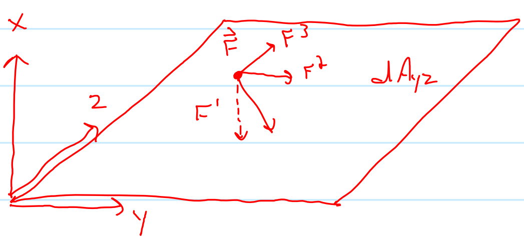
Consider: $n_\mu = (0,1,0,0)$ $dV = dt\,dy\,dz = dt\,dA_{yz}$

Then: $dP^\mu = T^{\mu\nu} n_\nu \, dV$

$= T^{\mu 1} n_1 \, dV$

$= T^{\mu 1} dt\, dA_{yz} \Rightarrow dP^i = T^{i1} dt\, dA_{yz}$

$\Downarrow$



$T^{i1} = \frac{dp^i}{dt} \frac{1}{dA_{yz}}$

$T^{i1} = F^i_{Net} \frac{1}{dA_{yz}}$

$\Downarrow$

T^{11} = pressure on dA_{yz}

T^{21}, T^{31} = shear on dA_{yz}

In general: $T^{\mu\nu} = \left(\begin{array}{c|c} \text{Energy Density} & \text{Momentum Density} \\ \hline \text{Momentum Density} & \text{Pressure, Shear / Shear} \end{array}\right)$

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Consider: $T^{\mu\nu} = \underbrace{\frac{dP^\mu}{n_\nu dV}}_{\text{W.T.F.?!}}$

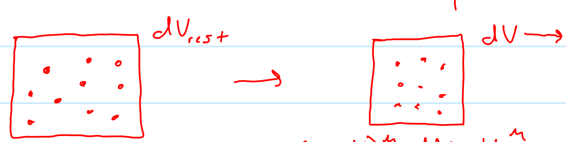
Recall: $dN^\mu = dn_{rest} U^\mu = dN \frac{U^\mu}{dV_{rest}}$

A very useful result for dividing by volumes

but $dN^\mu = \frac{dN}{n_\mu dV} \Rightarrow \frac{dN}{n_\mu dV} = dN \frac{U^\mu}{dV_{rest}} \Rightarrow \boxed{\frac{1}{n_\mu dV} = \frac{U^\mu}{dV_{rest}}} \Rightarrow T^{\mu\nu} = \frac{dP^\mu U^\nu}{dV_{rest}}$

Examples

Dust: Dust is defined as a collection of particles at rest w.r.t. each other.

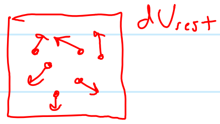


$\leftarrow dP^\mu = \underbrace{dN}_{\text{\# of particle}} \underbrace{m}_{\text{mass of each}} \underbrace{U^\mu}_{\text{4-velocity of each (same for all!)}}$

Then: $T^{\mu\nu} = \frac{dP^\mu U^\nu}{dV_{rest}} = \frac{dN m U^\mu U^\nu}{dV_{rest}} = dn_{rest}\, m\, U^\mu U^\nu = \underbrace{\rho}_{\text{rest energy density}} U^\mu U^\nu$

In the rest frame $U^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow T^{\mu\nu}_{rest} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Perfect Fluid: Collection of particles w/ random motion in overall rest frame.



Typically ignore viscosity (shear) so $T^{ij} = 0$ $i \neq j$
and assume isotropy so $T^{11} = T^{22} = T^{33} \equiv p$

no momentum density since there is no net flow

Then $T^{\mu\nu}_{rest} = \begin{pmatrix} \rho & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{pmatrix}$

A clever (and useful) way to write this is as: $T^{\mu\nu} = (\rho + p) U^\mu U^\nu + p\eta^{\mu\nu}$

To check remember $U^\mu_{rest} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow T^{\mu\nu}_{rest} = (\rho + p)\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + p\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

$= \begin{pmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$ ✓

However: Since $T^{\mu\nu} = (\rho + p) U^\mu U^\nu + p\eta^{\mu\nu}$ is in terms of tensors, it must be true in <u>any</u> frame! Even when $U^\mu \neq \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

Another check: Dust has $p = 0 \Rightarrow T^{\mu\nu} = \rho U^\mu U^\nu$ ✓

In GR, $T^{\mu\nu}$ will play the role of sources for curvature in a manner analogous to how J^μ (charge-current density) does for E&M.
You can already see how Einstein's Eqn. will be harder (2-rank tensor source!)

Monday, February 02, 2015 8:20 AM

Conservation Laws

Once we have an appropriate relativistic density, conservation laws are simple:

N^μ number-current density $\quad \frac{\partial N^\mu}{\partial x^\mu} = 0 = \frac{\partial N}{\partial t} + \vec{\nabla} \cdot \vec{N}$

J^μ charge-current density $\quad \frac{\partial J^\mu}{\partial x^\mu} = 0 = \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J}$

$T^{\mu\nu}$ energy-momentum density $\quad \frac{\partial T^{\mu\nu}}{\partial x^\mu} = 0 \Rightarrow \begin{cases} \frac{\partial T^{\mu 0}}{\partial x^\mu} = 0 = \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{\pi} \\ \frac{\partial T^{\mu i}}{\partial x^\mu} = 0 \quad \text{relativistic fluid dynamics} \end{cases}$

That is, in all case you can consider a 3D volume in $\mathbb{M}^4$ and the flux through the bounding surface (using Stoke's) is the time-rate change of the total number, charge, energy, etc.

Tuesday, February 03, 2015 12:09 PM

These expressions are referring to one component at a time.

A few things to keep in mind:

3D Galilean Spacetime	$g_{ij} = \delta_{ij}$	$\delta_{ij} = \delta^{ij}$	$V^i = V_i$	$x^i(t)$
4D Minkowski Spacetime	$g_{\mu\nu} = \eta_{\mu\nu}$	$\eta_{\mu\nu} = \eta^{\mu\nu}$	$V^\mu \neq V_\mu$	$x^\mu(\tau)$
4D Curved Spacetime	$\underbrace{g_{\mu\nu}(x^\mu)}_{\text{The metric is a}}$	$\underbrace{g_{\mu\nu} \neq g^{\mu\nu}}_{\text{careful!}}$	$V^\mu \neq V_\mu$	$x^\mu(\tau)$

function to be determined by solving Einstein's Equation

Tuesday, February 03, 2015 12:40 PM

Do we really have to curve spacetime? Why can't we just "relativize" Newtonian gravity?

Newton: $\vec{F}_{G_{12}} = \frac{G m_1 m_2}{|\vec{r}_1(t) - \vec{r}_2(t)|^3} \left(\vec{r}_1(t) - \vec{r}_2(t) \right)$

This expression uses the same t for $\vec{r}_1$ and $\vec{r}_2$ which implies action-at-a-distance or that gravitational effects propagate ∞-fast.
If we keep this same form and just put in a finite speed $\Rightarrow$ unstable orbits.
So naive relativistic gravity fails.

But we know a very similar theory that does work w/ finite speed:

Coulomb: $\vec{F}_{C_{12}} = \frac{k q_1 q_2}{|\vec{r}_1(t) - \vec{r}_2(t)|^3} \left(\vec{r}_1(t) - \vec{r}_2(t) \right)$

This has the same naive problem as above, but the full story w/ B-fields, LW potentials, etc. works out consistently.

So perhaps we should relativize gravity by completing the analogy w/ E&M. We can do this and it will lead to gravito-magnetic effects (tiny because you need big M for gravity but this usually forces $\frac{v}{c} \ll 1$.

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There are two big problems with this type of relativistic gravity:

i) It is still based on m and so misses the observed effects on $m=0$ (gravitational lensing)

ii) It doesn't even get the predictions right when $m \neq 0$ (though it will serve as a good approximation to GR in the weak-field limit).

The root of the problem is that we need to think more deeply about $m_I = m_G$. In doing so we will eventually get to the fundamental physical principle underlying GR.